Euclidean Spaces

A vector space $\mathbb{E}$ is called a Euclidean Space if it has a SCALAR PRODUCT defined in it.

$$\mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}$$

$$\langle \bar{u}, \bar{v} \rangle = \mu \in \mathbb{R}$$

<u>SYMMETRY</u> $\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$

<u>LINEALITY</u> $\langle \alpha \bar{u} + \beta \bar{v}, \bar{w} \rangle = \alpha \langle \bar{u}, \bar{w} \rangle + \beta \langle \bar{v}, \bar{w} \rangle$

<u>POSITIVITY</u> $\langle \bar{u}, \bar{u} \rangle > 0$

Scalar Product in $\mathbb{E}$

Example: Prove $\forall p(x), q(x) \in \mathbb{P}_2$ $\boxed{\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx}$ is a scalar product of $\mathbb{P}_2$.

① Symmetry $\langle p(x), q(x)\rangle = \int_0^1 p(x)\, q(x)\, dx = \int_0^1 q(x)\, p(x)\, dx = \langle q(x), p(x)\rangle$ ✓

② Linearity $\langle \alpha p(x) + \beta q(x), t(x)\rangle = \int_0^1 (\alpha p(x) + \beta q(x))\, t(x)\, dx = \int_0^1 (\alpha p(x) t(x) + \beta q(x) t(x))\, dx =$

$= \alpha \int_0^1 p(x) t(x) dx + \beta \int_0^1 q(x) t(x) dx = \alpha \langle p(x), t(x)\rangle + \beta \langle q(x), t(x)\rangle$ ✓

③ Positivity $\langle p(x), p(x)\rangle = \int_0^1 p(x) p(x) dx = \int_0^1 \underbrace{[p(x)]^2}_{>0} dx > 0$ ✓

$\boxed{\text{Yes, it is a Scalar Product}}$

How it works:

$p(x) = x^2$

$q(x) = x + 1$

$\langle p(x), q(x)\rangle = \int_0^1 x^2(x+1)dx = \left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12} \in \mathbb{R}$

Gramm's matrix

In $\mathbb{R}^3(\mathbb{R})$ with a base $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

$$\langle \bar{u}, \bar{v} \rangle = \underbrace{\mu}_{\in \mathbb{R}} \iff \underbrace{(u^1\ u^2\ u^3)_B}_{\bar{u}} \underbrace{\begin{pmatrix} \langle \bar{e}_1, \bar{e}_1 \rangle & \langle \bar{e}_1, \bar{e}_2 \rangle & \langle \bar{e}_1, \bar{e}_3 \rangle \\ \langle \bar{e}_1, \bar{e}_2 \rangle & \langle \bar{e}_2, \bar{e}_2 \rangle & \langle \bar{e}_2, \bar{e}_3 \rangle \\ \langle \bar{e}_1, \bar{e}_3 \rangle & \langle \bar{e}_2, \bar{e}_3 \rangle & \langle \bar{e}_3, \bar{e}_3 \rangle \end{pmatrix}}_{G_B \ \equiv \text{Gramm's Matrix}} \underbrace{\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}_B}_{\bar{v}} = \underbrace{\mu}_{\in \mathbb{R}}$$

e.g. Build the Gramm Matrix of the previous example for base $B = \{x^2, x, 1\}$

$$G_B = \begin{pmatrix} \langle x^2, x^2 \rangle & \langle x^2, x \rangle & \langle x^2, 1 \rangle \\ \langle x^2, x \rangle & \langle x, x \rangle & \langle x, 1 \rangle \\ \langle x^2, 1 \rangle & \langle x, 1 \rangle & \langle 1, 1 \rangle \end{pmatrix} = \begin{pmatrix} 1/5 & 1/4 & 1/3 \\ 1/4 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1 \end{pmatrix}$$

$$\langle x^2, x^2 \rangle = \int_0^1 x^2 x^2 dx = \frac{1}{5} \qquad \langle x, x \rangle = \frac{1}{3}$$

$$\langle x^2, x \rangle = \int_0^1 x^2 x \, dx = \frac{1}{4} \qquad \langle x, 1 \rangle = \frac{1}{2}$$

$$\langle x^2, 1 \rangle = \int_0^1 x^2 \cdot 1 \, dx = \frac{1}{3} \qquad \langle 1, 1 \rangle = 1$$

How it works:

$p(x) = x^2$

$q(x) = x + 1$

$$\langle p(x), q(x) \rangle = \int_0^1 x^2(x+1)dx = \frac{7}{12} \in \mathbb{R}$$

$$\langle p(x), q(x) \rangle = \overbrace{\underbrace{(1\ 0\ 0)_B}_{p(x)} G_B}^{(1/5\ 1/4\ 1/3)} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}_B}_{q(x)} = (1/5\ 1/4\ 1/3) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

ORTHOGONALITY $\bar{u}$ and $\bar{v}$ are ORTOGONAL $\iff \langle \bar{u}, \bar{v} \rangle = 0$

MODULUS $|\bar{u}| = \sqrt{\langle \bar{u}, \bar{u} \rangle}$ Modulus of $\bar{u}$

ANGLES $\langle \bar{u}, \bar{v} \rangle = |\bar{u}||\bar{v}| \cos(\widehat{\bar{u}, \bar{v}}) \longrightarrow (\widehat{\bar{u}, \bar{v}}) = \arccos \dfrac{\langle \bar{u}, \bar{v} \rangle}{|\bar{u}||\bar{v}|}$ ≡ angle formed by $\bar{u}$ and $\bar{v}$

Orthogonal Subspace ($\omega(L)$ or $L^{\perp}$)

Given L, a subspace of $\mathbb{E}$, $\omega(L)$ is the orthogonal subspace to L, where:

$$\omega(L) = \{ \bar{x} \in \mathbb{E} \,/\, \langle \bar{x}, \bar{y} \rangle = 0 \quad \forall \bar{y} \in L \}$$

Orthogonality and Gramm

$$G_B = \begin{pmatrix} \underbrace{\langle \bar{e}_1, \bar{e}_1 \rangle}_{|\bar{e}_1|^2} & \langle \bar{e}_1, \bar{e}_2 \rangle & \langle \bar{e}_1, \bar{e}_3 \rangle \\ \langle \bar{e}_1, \bar{e}_2 \rangle & \underbrace{\langle \bar{e}_2, \bar{e}_2 \rangle}_{|\bar{e}_2|^2} & \langle \bar{e}_2, \bar{e}_3 \rangle \\ \langle \bar{e}_1, \bar{e}_3 \rangle & \langle \bar{e}_2, \bar{e}_3 \rangle & \underbrace{\langle \bar{e}_3, \bar{e}_3 \rangle}_{|\bar{e}_3|^2} \end{pmatrix}$$

A vector is NORMAL when $|\bar{u}| = 1$

B is an ORTHOGONAL base $\iff \left\{ \begin{array}{l} \langle \bar{e}_1, \bar{e}_2 \rangle = 0 \\ \langle \bar{e}_1, \bar{e}_3 \rangle = 0 \\ \langle \bar{e}_2, \bar{e}_3 \rangle = 0 \end{array} \right\} \iff G_B = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$

B is an ORTHONORMAL base $\iff \left\{ \begin{array}{l} \langle \bar{e}_1, \bar{e}_2 \rangle = 0 \\ \langle \bar{e}_1, \bar{e}_3 \rangle = 0 \\ \langle \bar{e}_2, \bar{e}_3 \rangle = 0 \\ |\bar{e}_1| = |\bar{e}_2| = |\bar{e}_3| = 1 \end{array} \right\} \iff G_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

The usual scalar product (orthonormal base)

$\bar{x} = (x^1, x^2, x^3)_B$

$\bar{y} = (y^1, y^2, y^3)_B$

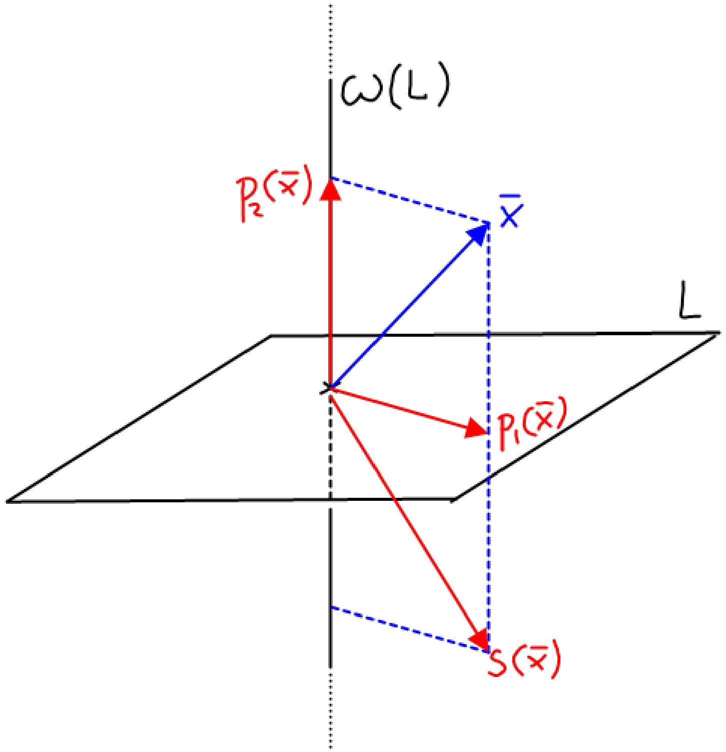
$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ an ORTHONORMAL Base of $\mathbb{R}^3$

$$\boxed{\begin{array}{ll} \langle \bar{e}_1, \bar{e}_2 \rangle = 0 & |\bar{e}_1| = 1 \\ \langle \bar{e}_1, \bar{e}_3 \rangle = 0 & |\bar{e}_2| = 1 \\ \langle \bar{e}_2, \bar{e}_3 \rangle = 0 & |\bar{e}_3| = 1 \end{array}} \Leftrightarrow G_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$\langle \bar{x}, \bar{y} \rangle = \overbrace{(x^1 \; x^2 \; x^3)_B}^{\bar{x}} \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{G_B} \overbrace{\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}_B}^{\bar{y}} =$$

$$= x^1 y^1 + x^2 y^2 + x^3 y^3$$

Projection of a vector $\bar{x}$ on L and $\omega(L)$



Since $\boxed{\omega(L) \oplus L}$ ALWAYS

$\forall \bar{x} \in \mathbb{E} \quad \bar{x} = P_1(\bar{x}) + P_2(\bar{x})$